On the Incompleteness of Berger's List of Holonomy Representations

Quo-Shin Chi*
Sergey A. Merkulov
Lorenz J. Schwachhöfer

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Abstract

In 1955, Berger [Ber] gave a list of irreducible reductive representations which can occur as the holonomy of a torsion-free affine connection. This list was stated to be complete up to possibly a finite number of missing entries.

In this paper, we show that there is, in fact, an infinite family of representations which are missing from this list, thereby showing the incompleteness of Berger's classification. Moreover, we develop a method to construct torsion-free connections with prescribed holonomy, and use it to give a complete description of the torsion-free affine connections with these new holonomies. We also deduce some striking facts about their global behaviour.

1 Introduction

An affine connection is one of the basic objects of interest in differential geometry. It provides a simple and invariant way of transferring information from one point of a connected manifold M to another and, not surprisingly, enjoys lots of applications in many branches of mathematics, physics and mechanics. Among the most informative characteristics of an affine connection is its (restricted) holonomy group which is defined, up to conjugacy, as the subgroup of $Gl(T_tM)$ consisting of all automorphisms of the tangent space T_tM at $t \in M$ induced by parallel translations along t-based loops in M.

Which reductive Lie groups G can be irreducibly acting holonomies of affine connections?

By a result of Hano and Ozeki [HO], *any* (closed) Lie group representation $G \subseteq Gl(V)$ can be realized in this way. The same question, if posed in the subclass of *torsion-free* affine connections, has a very different answer. Long ago, Berger [Ber] presented a very restricted list of possible holonomies of a torsion-free affine connection which, as he claimed, is complete up to a finite number of missing terms. His list is separated into

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two parts. The first part corresponds to the holonomies of *metric* connections, the second part to the *non-metric* ones. While Berger gave detailed arguments for the proof of the metric part, the proof of the second part was omitted.

The list of metric connections has been studied extensively in the intervening years. In fact, it is by now well-known which entries of this list actually *do* occur as holonomies of torsion-free connections, and how the holonomy relates to both the geometry and the topology of the underlying manifold. See [A], [Br1], [Br2], [J], [Si], as well as the surveys in [Sa] and [Bes].

Despite the lack of proof, the second part of Berger's holonomy list seems to have been generally accepted as correct. Even when Bryant found examples of holonomies which are not on Berger's list [Br3], he called them *exotic holonomies*, suggesting that such holonomies should exist in very special dimensions only, and therefore should be thought of as analogous to the *exceptional holonomies* in the metric case. Further examples of exotic holonomies were found in [CS].

However, it is the subject of the present article to prove that, even up to finitely many missing terms, Berger's list is still incomplete. This is done by proving the existence of an infinite family of irreducible representations which are not on this list, yet do occur as holonomy of torsion-free connections. These representations are:

$$\begin{array}{llll} Sl(2,\mathbb{C})SO(n,\mathbb{C}), & \text{acting on} & \mathbb{R}^{8n} \cong \mathbb{C}^2 \otimes \mathbb{C}^n, & \text{where } n \geq 3, \\ Sl(2,\mathbb{R})SO(p,q), & \text{acting on} & \mathbb{R}^{2(p+q)} \cong \mathbb{R}^2 \otimes \mathbb{R}^{p+q}, & \text{where } p+q \geq 3, \\ Sl(2,\mathbb{R})SO(2,\mathbb{R}), & \text{acting on} & \mathbb{R}^4 \cong \mathbb{R}^2 \otimes \mathbb{R}^2. & \end{array} \tag{1}$$

Theorem 1.1 *All representations in* (1) *occur as holonomies of torsion-free connections which are not locally symmetric.*

These candidates were first discovered by twistor theoretical methods which shall be described in section 2. In fact, the formal existence of holomorphic torsion-free connections whose holonomy is listed in the first family of (1) can be shown, in principle, by deformation theory only.

We shall, however, present a different, technically simple method to prove their existence in section 3. This new approach will allow us to treat all representations in (1) simultaneously, and to assert some global geometric properties and rigidity of connections with these holonomies which seem hard to achieve otherwise. In fact, we shall classify *all* connections with the above holonomies by showing that they all arise from this construction. This implies, for example, the following:

Theorem 1.2 *Let* $G \subseteq Gl(V)$ *be one of the representations in* (1) *other than* $Sl(2,\mathbb{R})SO(2,\mathbb{R})$. *Then the following hold.*

(1) *Any torsion-free connection whose holonomy is contained in* G *is analytic.*

(2) *The moduli space of torsion-free connections whose holonomy is contained in G is finite dimensional. In particular, the 2nd derivative of the curvature at a single point completely determines the connection.*

(3) *If* $n \equiv 0, 1 \bmod 4$ *then any torsion-free connection whose holonomy is contained in G admits a non-trivial infinitesimal symmetry, i.e. a vector field whose flow preserves the connection.*

(4) *If $G \neq Sl(2,\mathbb{R})SO(n,\mathbb{R})$, then a torsion-free connection whose holonomy is contained in G is geodesically incomplete, unless the connection is locally symmetric. If the latter is the case, then the holonomy is a proper subgroup of G.*

The final step towards the proof Theorem 1.1 was accomplished when the second and third author met at the conference *Geometry and Physics* in Aarhus, Denmark. We are grateful to the organizers of this conference for inviting us; the third author wishes to thank Professor Friedrich Hirzebruch for providing financial support for his participation in that conference.

2 Connections and Legendre moduli spaces

Let $G \subseteq Gl(V)$ be an effective irreducible representation of a complex connected reductive Lie group G on the finite dimensional complex vector space V. Without further comment, we shall always regard G as a Lie group with a fixed representation on V. Clearly, G also acts on V^* via the dual representation. Let $\tilde{X}$ be the G-orbit of a highest weight vector in V^*. Then the quotient $X := \tilde{X}/\mathbb{C}^*$ is a generalized flag variety which is canonically embedded into $\mathbb{P}(V^*)$. In fact, $X = G_s/P$, where G_s is the semi-simple part of G, and where P is the parabolic subgroup of G_s leaving the highest weight vector invariant up to a scalar. Let L_X be the restriction of the hyperplane bundle $\mathcal{O}(1)$ on $\mathbb{P}(V^*)$ to the submanifold $X \hookrightarrow \mathbb{P}(V^*)$. Then $V \cong H^0(X, L_X)$, and the Lie algebra $\mathfrak{g}$ of G is contained in the Lie algebra

$$H^0\left(X, L_X \otimes \left(J^1 L_X\right)^*\right) \cong H^0(X, TX) \oplus \mathbb{C}.$$

Therefore, to the representation $G \subseteq Gl(V)$, we canonically associate a pair (X, L_X), consisting of a generalized flag variety X and an ample line bundle L_X on X. Most of the relevant information about G can be restored from (X, L_X). The very few cases when important information gets lost by the transition from (X, L_X) back to $G \subseteq Gl(V)$ are listed in [St].

A crucial step in deciding whether G can occur as the holonomy of a torsion-free affine connection is the computation of the formal curvature space $K(\mathfrak{g})$, which is defined as the kernel of the composition

$$\Lambda^2 V^* \otimes \mathfrak{g} \to \Lambda^2 V^* \otimes V^* \otimes V \to \Lambda^3 V^* \otimes V.$$

Namely, it follows from the *Ambrose-Singer Holonomy Theorem* [AS] that if there is a *proper* subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ such that $K(\mathfrak{g}) \subseteq \Lambda^2 V^* \otimes \mathfrak{g}' \subset \Lambda^2 V^* \otimes \mathfrak{g}$, then G *cannot* be

holonomy of a torsion-free connection. This condition is called *Berger's first criterion*, and was used for the original classification in [Ber].

It is desirable to interpret $K(\mathfrak{g})$ in terms of the associated pair (X, L_X), because one will then be able to make use of the powerful tools of complex analysis (such as, say, the vanishing theorems) to attack the Berger classification problem. The twistor theory developped in [M] does indeed provide us with such an interpretation. We shall briefly describe this twistor construction, and use it to calculate $K(\mathfrak{g})$ for the representations in (1) explicitly. For details and more general statements, we refer to [M].

Let Y be a complex $(2n+1)$-dimensional manifold. A *complex contact structure* on Y is a rank $2n$ holomorphic subbundle $D \subseteq TY$ of the holomorphic tangent bundle to Y such that the Frobenius form

$$\begin{array}{rcl} \Phi : D \times D & \longrightarrow & TY/D \\ (v, w) & \longmapsto & [v, w] \bmod D \end{array}$$

is non-degenerate. Define the contact line bundle L by the exact sequence

$$0 \longrightarrow D \longrightarrow TY \xrightarrow{\theta} L \longrightarrow 0.$$

One can easily verify that the maximal non-degeneracy of the distribution D is equivalent to the fact that the above defined "twisted" 1-form $\theta \in H^0(Y, L \otimes \Omega^1 M)$ satisfies the condition

$$\theta \wedge (d\theta)^n \neq 0.$$

A complex n-dimensional submanifold X of the complex contact manifold Y is called a *Legendre submanifold* if $TX \subseteq D$.

Suppose $X \hookrightarrow Y$ is a compact complex Legendre submanifold of a complex contact manifold (Y, L), and denote the restriction $L|_X$ by L_X. If $H^1(X, L_X) = 0$, then there exists a complete family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact complex Legendre submanifolds which is obtained from X by all possible holomorphic Legendre deformations of X inside Y. The parameter space M is an $h^0(X, L_X)$-dimensional complex manifold, called the *Legendre moduli space*. Moreover, if two more cohomology groups vanish, namely if

$$H^0\left(X, L_X \otimes S^2\left(J^1 L_X\right)^*\right) = H^1\left(X, L_X \otimes S^2\left(J^1 L_X\right)^*\right) = 0, \tag{2}$$

then the Legendre moduli space M comes equipped not only with an induced complex manifold structure, but also with a uniquely induced holomorphic torsion-free affine connection whose curvature tensor is a field on M with values in the cohomology group $H^1\left(X, L_X \otimes S^3\left(J^1 L_X\right)^*\right)$. In fact, *any* holomorphic torsion-free affine connection with reductive irreducibly acting holonomy group is an induced connection on an appropriate Legendre moduli space [M].

This suggests the following strategy to look for new exotic holonomies: find a pair (X, L_X) consisting of a generalized flag variety $X = G/P$ and an ample line bundle $L_X \to X$ such that the cohomology groups in (2) vanish, while $H^1\left(X, L_X \otimes S^3\left(J^1 L_X\right)^*\right)$ is non-zero.

Then the twistor theory guarantees that there is a natural injection

$$\imath : H^1\left(X, L_X \otimes S^3\left(J^1 L_X\right)^*\right) \longrightarrow \Lambda^2 V^* \otimes \mathfrak{g}, \tag{3}$$

where $\mathfrak{g} := H^0\left(X, L_X \otimes \left(J^1 L_X\right)^*\right)$ and $V := H^0(X, L_X)$, whose image equals $K(\mathfrak{g})$. In particular, $K(\mathfrak{g}) \cong H^1\left(X, L_X \otimes S^3\left(J^1 L_X\right)^*\right)$ as a $\mathfrak{g}$-vector space. We let $K_0(\mathfrak{g}) \subseteq K(\mathfrak{g})$ be the set of elements with *full curvature*, i.e.

$$K_0(\mathfrak{g}) := \{\rho \in K(\mathfrak{g}) \mid \langle\{\rho(x,y) \mid x, y \in V\}\rangle = \mathfrak{g}\}. \tag{4}$$

One can show that either $K_0(\mathfrak{g}) = \emptyset$ or $K_0(\mathfrak{g})$ is *dense in* $K(\mathfrak{g})$.

As a particular example, let us consider the reducible homogeneous manifold $X = \mathbb{CP}_1 \times \mathbb{Q}_m$, where $\mathbb{Q}_m \hookrightarrow \mathbb{CP}_{m+1}$ is the non-degenerate quadric. We define the line bundle

$$L_X := \pi_1^*(\mathcal{O}_{\mathbb{CP}_1}(1)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{CP}_{m+1}}(1)|_{\mathbb{Q}_m}),$$

where $\pi_1 : X \to \mathbb{CP}_1$ and $\pi_2 : X \to \mathbb{Q}_m$ are the canonical projections. Then

$$V := H^0(X, L_X) = W_2 \otimes W_n,$$

where $W_2 := H^0(\mathbb{CP}_1, \mathcal{O}_{\mathbb{CP}_1}(1))$ and $W_n := H^0(\mathbb{Q}_m, \mathcal{O}_{\mathbb{CP}_{m+1}}(1)|_{\mathbb{Q}_m})$ are vector spaces of dimensions 2 and $n := m + 2$, respectively, and

$$\mathfrak{g} := H^0\left(X, L_X \otimes \left(J^1 L_X\right)^*\right) = \mathfrak{g}_0 \oplus \mathbb{C},$$

where $\mathfrak{g}_0 = \mathfrak{sl}(W_2) \oplus \mathfrak{so}(W_n)$. Note that W_2 and W_n carry a $\mathfrak{g}_0$-invariant area form $<\,,\,>$ and a $\mathfrak{g}_0$-invariant inner product $(\ ,\ )$, respectively. It is easy to check that for $m \geq 2$, i.e. $n \geq 4$, the cohomology groups in (2) vanish, while

$$K(\mathfrak{g}) \cong H^1\left(X, L_X \otimes S^3\left(J^1 L_X\right)^*\right) \cong S^2(W_2) \oplus \Lambda^2 W_n. \tag{5}$$

A calculation shows that $\Lambda^2 V^* \otimes \mathfrak{g}$ contains only two summands isomorphic to $S^2(W_2)$ and three isomorphic to $\Lambda^2 W_n$. After that, it is easy to obtain explicit expressions for the elements of $K(\mathfrak{g})$. As it turns out, all elements of $K(\mathfrak{g})$ are contained in $\Lambda^2 \otimes \mathfrak{g}_0 \subset \Lambda^2 \otimes \mathfrak{g}$. It follows that *every torsion-free connection whose holonomy is contained in* $Gl(2,\mathbb{C})SO(n,\mathbb{C})$ *for* $n \geq 3$ *must be contained in* $Sl(2,\mathbb{C})SO(n,\mathbb{C})$.

In order to present the description of $K(\mathfrak{g})$, we use the identifications $\mathfrak{sl}(W_2) \cong S^2(W_2)$ and $\mathfrak{so}(W_n) \cong \Lambda^2 W_n$, given by the identities

$$\begin{array}{rcrcl} (e_1 e_2) \cdot e_3 & := & <e_1, e_3> e_2 & + & <e_2, e_3> e_1, \quad \text{and} \\ (x_1 \wedge x_2) \cdot x_3 & := & (x_1, x_3) x_2 & - & (x_2, x_3) x_1 \end{array} \tag{6}$$

for all $e_i \in W_2$, $x_i \in W_n$.

Proposition 2.1 *Let* $G \subseteq Gl(V)$ *be one of the representations in (1) other than* $Sl(2,\mathbb{R})SO(2,\mathbb{R})$*, and let* $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ *be its Lie algebra. For* $A \in \mathfrak{sl}(2,\mathbb{C})$ *and* $M \in \mathfrak{so}(n,\mathbb{C})$ *(*$A \in \mathfrak{sl}(2,\mathbb{R})$ *and* $M \in \mathfrak{so}(p,q)$*, respectively), define* $\rho_{A+M} : \Lambda^2 V \to \mathfrak{g}$ *by*

$$\begin{aligned} \rho_{A+M}(e_1 \otimes x_1, e_2 \otimes x_2) := \; & <e_1, e_2> ((x_1, x_2)(A + M) \; + \; (x_1 \wedge M x_2 + x_2 \wedge M x_1)) \\ & + (x_1, M x_2) e_1 e_2 \; - \; <A e_1, e_2> x_1 \wedge x_2. \end{aligned} \tag{7}$$

Then $\rho_{A+M} \in K(\mathfrak{g})$, and the map

$$\begin{array}{rccc} \rho: & \mathfrak{g} & \longrightarrow & K(\mathfrak{g}) \\ & A+M & \longmapsto & \rho_{A+M} \end{array}$$

is a G-equivariant isomorphism.

Proof. It is straightforward to verify that $\rho_{A+M} \in K(\mathfrak{g})$ and that ρ is a G-equivariant injection. If $G = Sl(2,\mathbb{C})SO(n,\mathbb{C})$ with $n \geq 4$, then the surjectivity of ρ follows from (5). This implies the surjectivity of ρ for $G = Sl(2,\mathbb{R})SO(p,q)$ with $p+q \geq 4$ as well.

If $n = 3$ or $p + q = 3$, then the surjectivity follows from a direct calculation. ∎

Remark 2.2 (1) If $G = Sl(2,\mathbb{R})SO(2,\mathbb{R})$ then ρ is injective but not surjective. In fact, $dim(K(\mathfrak{g})) = 9 > dim(\mathfrak{g})$ in this case.

(2) The case $G = Sl(2,\mathbb{F})SO(3,\mathbb{F})$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ was treated in [CS], where this representation was called H_{12}.

Note that the twistor approach from above does not work for $n = 3$; indeed, the cohomology $H^1\left(X, L_X \otimes S^2\left(J^1 L_X\right)^*\right) \neq 0$, thus (2) is *not* satisfied. Nevertheless, it follows from the results in [CS] that every Legendre moduli space with $X \cong \mathbb{CP}_1 \times \mathbb{Q}_1 \cong \mathbb{CP}_1 \times \mathbb{CP}_1$ and L_X as above, i.e. $L_X \cong \mathcal{O}(1,2)$, *does* carry an induced torsion-free connection. This shows, in particular, that (2) is not a necessary condition for the existence of torsion-free induced connections on a Legendre moduli space (cf. [M]).

(3) If $A, M \neq 0$ then ρ_{A+M} is surjective. Thus, for the representations in (1), $K_0(\mathfrak{g}) \subseteq K(\mathfrak{g})$ is open dense.

By (3), it follows that if there exists a holomorphic torsion-free affine connection ∇ with "generic" curvature tensor, then its holonomy will be the full group $Sl(2,\mathbb{C})SO(n,\mathbb{C})$.

If (Y, L) is a complex contact manifold containing $X = \mathbb{CP}_1 \times \mathbb{Q}_m$ as a Legendre submanifold, then the latter is stable under arbitrary holomorphic deformations of the contact structure on Y. Therefore, one possible way to prove the existence of the required holomorphic affine connections is to use the Kodaira deformation theory [K] to construct a sufficiently "generic" contact manifold Y.

We shall, however, use a different method for the existence proof, which will be presented in the following section.

3 Construction of torsion-free connections

Let us briefly recall the definition and basic properties of a Poisson manifold. For a more detailed exposition, see e.g. [LM] or [V].

Definition 3.1 *A Poisson structure on a differentiable manifold P is a bilinear map, called the Poisson bracket*

$$\{\ ,\ \} : \otimes^2 C^\infty(P, \mathbb{R}) \longrightarrow C^\infty(P, \mathbb{R}),$$

satisfying the following identities:

(i) *the bracket is skew-symmetric:*

$$\{f, g\} = -\{g, f\},$$

(ii) *the bracket satisfies the Jacobi identity:*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,$$

(iii) *the bracket is a derivation in each of its arguments:*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

It is well-known that on every Poisson manifold $(P, \{\ ,\ \})$, there exists a unique smooth bivector field $\Lambda \in \Gamma(P, \Lambda^2 TP)$ such that the Poisson bracket is given by

$$\{f, g\} = \Lambda(df, dg). \tag{8}$$

We define the homomorphism $\Lambda^\# : T^*P \to TP$ by the equation

$$(\Lambda^\# df)(g) = \{f, g\} \quad \text{for all } f, g \in C^\infty(P, \mathbb{R}). \tag{9}$$

The *half-rank at* $p \in P$ of the Poisson structure is the smallest integer r such that

$$\Lambda^{r+1}(p) = 0,$$

and the *rank at* $p \in P$ is twice the half-rank. It follows that the rank at P equals the rank of $\Lambda^\#_p : T^*_p P \to T_p P$. The Poisson structure is called *non-degenerate at* p if $\Lambda^\#_p$ is an isomorphism, i.e. if the rank at p equals the dimension of P. In particular, if P is non-degenerate at a point then P must be even dimensional, and the set of non-degenerate points is open in P. If P is non-degenerate *everywhere*, then there is a natural symplectic 2-form σ on P such that $\Lambda^\#$ is precisely the index-raising map associated to σ. In fact, it is well known that symplectic structures are in a natural one-to-one correspondence with non-degenerate Poisson structures.

The *characteristic field* of the Poisson structure is the subset of TP given by

$$\mathcal{C} = \Lambda^\#(T^*P).$$

Thus, the dimension of $\mathcal{C}_p$ equals the rank at p. A *characteristic leaf* $\Sigma \subseteq P$ is a submanifold for which $T_p\Sigma = \mathcal{C}_p$ for all $p \in \Sigma$. From (9), it follows that the set of functions which vanish on Σ form a *Poisson ideal;* hence there is a naturally induced Poisson structure on Σ. Clearly, this Poisson structure on Σ is non-degenerate. Thus it follows that *every characteristic leaf of a Poisson manifold carries a natural symplectic structure.*

Definition 3.2 *Let $(P, \{\ ,\ \})$ be a Poisson manifold. A symplectic realization of P is a symplectic manifold (S, σ) and a submersion*

$$\pi : S \longrightarrow P$$

which is compatible with the Poisson structures, i.e.

$$\{\pi^*(f), \pi^*(g)\}_S = \pi^*(\{f, g\}) \quad \textit{for all } f, g \in C^\infty(P, \mathbb{R}), \tag{10}$$

where the Poisson bracket $\{\ ,\ \}_S$ on S is induced by the symplectic structure.

Proposition 3.3 *If $p_0 \in P$ has locally constant rank, i.e. the rank is constant on an open neighborhood U of p_0, then there is a local symplectic realization at p_0, i.e. a symplectic realization $\pi : S \longrightarrow U'$ with $p_0 \in U' \subseteq U$.*

Proof. By Darboux's Theorem, there exists a local coordinate system (p_i, q_i, t_α), $i = 1, \ldots, r$, $\alpha = 1, \ldots, s$, in a neighborhood U of p_0 such that the Poisson bracket is given by

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}.$$

Let $S := U \times \mathbb{R}^s$ with coordinates u_α on $\mathbb{R}^s$. Define the symplectic 2-form $\sigma := dp_i \wedge dq_i + dt_\alpha \wedge du_\alpha$ on S. Then it is easily verified that the natural projection $\pi : S \to U$ is a symplectic realization of U. ∎

We now turn to the construction of torsion-free connections via Poisson structures. First of all, let us set up some notation.

Let V be a finite dimensional vector space, $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ a Lie sub-algebra, and let $G \subseteq Gl(V)$ be the corresponding connected Lie group. G acts canonically on V, and on $\mathfrak{g}$ via the adjoint representation. This induces G-actions on all tensor powers and direct sums of $\mathfrak{g}$ and V which we will call the *canonical action* of G on these spaces.

Recall that the *curvature space* $K(\mathfrak{g})$ is defined by the exact sequence

$$0 \longrightarrow K(\mathfrak{g}) \longrightarrow \Lambda^2 V^* \otimes \mathfrak{g} \longrightarrow \Lambda^3 V^* \otimes V,$$

where the latter map is the composition of the inclusion and the skew-symmetrization map $\Lambda^2 V^* \otimes \mathfrak{g} \to \Lambda^2 V^* \otimes V^* \otimes V \to \Lambda^3 V^* \otimes V$. Likewise, we define the *2nd curvature space* $K^1(\mathfrak{g})$ by the exact sequence

$$0 \longrightarrow K^1(\mathfrak{g}) \longrightarrow V^* \otimes K(\mathfrak{g}) \longrightarrow \Lambda^3 V^* \otimes \mathfrak{g},$$

where again, the latter map is given by the composition of an inclusion and skew-symmetrization, namely $V^* \otimes K(\mathfrak{g}) \to V^* \otimes \Lambda^2 V^* \otimes \mathfrak{g} \to \Lambda^3 V^* \otimes \mathfrak{g}$. In other words, $K(\mathfrak{g})$ and $K^1(\mathfrak{g})$ consist of those linear maps $\Lambda^2 V \to \mathfrak{g}$ and $V \to K(\mathfrak{g})$ which satisfy the 1st and 2nd Bianchi identity, respectively.

Let $W := \mathfrak{g} \oplus V$. Denote elements of $\mathfrak{g}$ and V by $A, B, \dots$ and $x, y, \dots$, respectively, and elements of W by $w, w', \dots$. We may regard W as the semi-direct product of Lie algebras, i.e. we define a Lie algebra structure on W by the equation

$$[A + x, B + y] := [A, B] + A \cdot y - B \cdot x.$$

It is well-known [LM] that this induces a natural Poisson structure on the dual space W^*. Now, we wish to perturb this Poisson structure. For this, we need the

Definition 3.4 *A C^∞-map $\phi : \mathfrak{g}^* \to \Lambda^2 V^*$ is called admissible if*

(i) *ϕ is G-equivariant,*

(ii) *$d\phi(p) \in K(\mathfrak{g}) \subseteq \Lambda^2 V^* \otimes \mathfrak{g}$ for all $p \in \mathfrak{g}^*$.*

In order for condition (ii) to make sense, we use the natural identification $T_p^*\mathfrak{g}^* \cong \mathfrak{g}$. Now, the following important observation is easily proven.

Proposition 3.5 *Let V, $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, W and $K(\mathfrak{g})$ as above, and let $\phi : \mathfrak{g}^* \to \Lambda^2 V^*$ be an admissible map. Let $\Phi := \phi \circ pr$, where $pr : W^* \to \mathfrak{g}^*$ is the natural projection. Then the following bracket on W^* is Poisson:*

$$\{f, g\}(p) := p([A + x, B + y]) + \Phi(p)(x, y). \tag{11}$$

Here, $df_p = A + x$ and $dg_p = B + y$ are the decompositions of $df_p, dg_p \in T_p^* W^* \cong W$.

Note that for $\phi = 0$, we simply obtain the Poisson structure induced by the Lie algebra structure on W.

Let us now consider a Poisson structure on W^* induced by an admissible map $\phi : \mathfrak{g}^* \to \Lambda^2 V^*$. Let $\pi : S \to U$ be a symplectic realization of the open subset $U \subseteq W^*$. For each $w \in W$, we define the vector fields

$$\eta_w := \Lambda^\#(w) \in \mathfrak{X}(W^*),$$

and

$$\xi_w := \#(\pi^*(w)) \in \mathfrak{X}(S),$$

where $w \in W \cong T^*W^*$ is regarded as a 1-form on W^*. Since π preserves the Poisson structure (10), we have

$$\pi_*(\xi_w) = \eta_w \quad \text{for all } w \in W. \tag{12}$$

In contrast to the map $w \mapsto \eta_w$, the map $w \mapsto \xi_w$ is *pointwise injective*. Thus, $\xi := \{\xi_w \mid w \in W\} \subseteq TS$ is a distribution on S whose rank equals the dimension of W. For the bracket relations, we compute

$$\begin{array}{lcll} [\xi_A, \xi_B] & = & \xi_{[A,B]} & \\ [\xi_A, \xi_x] & = & \xi_{A \cdot x} & \\ [\xi_x, \xi_y](s) & = & \xi_{d\Phi(p)(x,y)} & \text{where } p = \pi(s). \end{array} \tag{13}$$

This implies, of course, that the distribution ξ on S is *integrable*. Moreover, the first equation in (13) implies that the flow along the vector fields ξ_A induces a local G-action on S. Let $F \subseteq S$ be a maximal integral leaf of ξ. Clearly, F is G-invariant, and we can define a W-valued coframe $\theta + \omega$ on F, where θ and ω take values in V and $\mathfrak{g}$, respectively, by the equation

$$v_s = \xi_{(\omega+\theta)(v_s)}(s), \text{ all } v_s \in T_s F.$$

The equations dual to (13) then read

$$\begin{aligned} d\theta &= -\omega \wedge \theta \\ d\omega &= -\omega \wedge \omega - \pi^*(d\Phi) \circ (\theta \wedge \theta). \end{aligned} \tag{14}$$

Here, $d\Phi$ is regarded as a map with values in $K(\mathfrak{g}) \subseteq \Lambda^2 V^* \otimes \mathfrak{g}$.

After shrinking S and U if necessary, we may assume that $M := F/G$ is a *manifold*. From (14) it follows that there is a unique torsion-free connection on M and a unique immersion $\imath : F \hookrightarrow \mathfrak{F}_V$ into the V-valued coframe bundle $\mathfrak{F}_V$ of M such that $\theta = \imath^*(\underline{\theta})$ and $\omega = \imath^*(\underline{\omega})$, where $\underline{\theta}$ and $\underline{\omega}$ are the tautological and the connection 1-form on $\mathfrak{F}_V$, respectively. Clearly, the holonomy of this connection is contained in G.

Definition 3.6 *Let $\phi : \mathfrak{g}^* \to \Lambda^2 V^*$ be an admissible map. Then a torsion-free connection which is obtained from the above construction is called a Poisson connection induced by ϕ.*

We then get the following result.

Theorem 3.7 *Let V, $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ and $K(\mathfrak{g})$ be as before, and let $K_0(\mathfrak{g}) \subseteq K(\mathfrak{g})$ be as in (4). Consider an admissible map $\phi : \mathfrak{g}^* \to \Lambda^2 V^*$. Furthermore, suppose that*

$$\mathfrak{g}^* \supseteq U_0 := (d\phi)^{-1}(K_0(\mathfrak{g})) \neq \emptyset.$$

Then there exist Poisson connections induced by ϕ whose holonomy representations are equivalent to $\mathfrak{g}$. Moreover, if $\phi|_{U_0}$ is not affine, then not all of these connections are locally symmetric.

Proof. Let $U^{reg} \subseteq U_0 \oplus V^* \subseteq W^*$ be the subset of points for which the rank is locally constant. By upper semi-continuity of the rank, U^{reg} is open dense in $U_0 \oplus V^*$.

Now Proposition 3.3 implies that there are symplectic realizations $\pi : S \to U$, with open $U \subseteq U^{reg} \subseteq W^*$. Then the above construction produces Poisson connections induced by ϕ on some manifold $M = F/G$. By (4), (14) and the *Ambrose-Singer Holonomy Theorem* [AS], the holonomy of this connection equals $\mathfrak{g}$.

To show the last part, let us assume that *all* connections which arise in this way are locally symmetric. Let $w := (p, q) \in U^{reg}$ with $p \in U_0$, $q \in V^*$. Then we may choose the symplectic realization $\pi : S \to U$ and $F \subseteq U$ such that $w \in \pi(F)$. It is then easy to show by (14) that the corresponding connection on $M := F/G$ is locally symmetric iff $\mathfrak{L}_{\xi_x}(\pi^*(d\Phi)) = 0$ for all $x \in V$. By (12) and because π is a submersion, this is equivalent to $\mathfrak{L}_{\eta_x}(d\Phi) = 0$ for all $x \in V$, or $\mathfrak{L}_{pr_*(\eta_x)}(d\phi) = 0$ for all $x \in V$. But now an easy calculation shows that for all $A \in \mathfrak{g}$,

$$(pr_*(\eta_x)_w)(A) = -q(A \cdot x) = -\jmath(q \otimes x)(A),$$

where $\jmath : V^* \otimes V \to \mathfrak{g}^*$ is the natural projection. Thus, by our assumption, it follows that

$$\mathfrak{L}_{\jmath(q\otimes x)}(d\phi)_p = 0 \text{ for all } x \in V,\ (p,q) \in U^{reg}.$$

By density of U^{reg}, this equation holds for *all* $p \in U_0$, $q \in V^*$, and since $\jmath$ is surjective, it follows that

$$\mathfrak{L}_\alpha(d\phi)_p = 0 \text{ for all } \alpha \in \mathfrak{g}^*,\ p \in U_0,$$

i.e. $d\phi|_{U_0}$ is constant, hence $\phi|_{U_0}$ is affine. ∎

By Theorem 3.7 it will suffice to address the question of *existence* of admissible maps ϕ in order to construct connections with prescribed holonomy.

Define the k-th jet space of $\mathfrak{g}$ by

$$J_k(\mathfrak{g}) := \left(S^k(\mathfrak{g}) \otimes \Lambda^2 V^*\right) \cap \left(S^{k-1}(\mathfrak{g}) \otimes K(\mathfrak{g})\right),$$

where both are regarded as subspaces of $S^{k-1}(\mathfrak{g})\otimes\mathfrak{g}\otimes\Lambda^2V^*$. Suppose there is a G-invariant element $\phi_k \in J_k(\mathfrak{g})$. If we regard ϕ_k as a polynomial map of degree k, $\phi_k : g^* \to \Lambda^2 V^*$, then it follows that ϕ_k is admissible. Conversely, given an *analytic* map $\phi : \mathfrak{g}^* \to \Lambda^2 V^*$ with analytic expansion at $0 \in \mathfrak{g}^*$

$$\phi = \phi_0 + \phi_1 + \cdots,$$

then it is straightforward to show that ϕ is admissible iff all ϕ_k are, iff $\phi_k \in J_k(\mathfrak{g})^G$.

Consider an element $\phi_2 \in J_2(\mathfrak{g})^G$. On the one hand, we may regard ϕ_2 as an element of $\mathfrak{g}\otimes K(\mathfrak{g})$, on the other hand, it is easy to verify that also $\phi_2 \in V \otimes K^1(\mathfrak{g}) \subseteq V \otimes V^* \otimes K(\mathfrak{g})$. Thus, by the natural contractions, ϕ_2 induces G-equivariant linear maps

$$\begin{array}{llll} \phi_2' : & \mathfrak{g}^* & \longrightarrow & K(\mathfrak{g}) \\ \phi_2'' : & V^* & \longrightarrow & K^1(\mathfrak{g}). \end{array} \tag{15}$$

We shall now demonstrate the existence of torsion-free connections with prescribed holonomy.

Theorem 3.8 *Let $G \subseteq Gl(V)$ be one of the following representations:*

(i) *For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $G = Sl(2,\mathbb{F})$ acting on the space $V_3 \subseteq \mathbb{F}[x,y]$ of homogeneous polynomials in x and y of degree 3,*

(ii) *any of the representations in* (1).

Then there is a G-invariant 2-form $\sigma \in \Lambda^2 V^*$ which is unique up to a scalar. Also, $J_2(\mathfrak{g})$ is one-dimensional and acted on trivially by G. The generic Poisson connection induced by the admissible map

$$\phi = \phi_2 + c\sigma, \tag{16}$$

with $0 \neq \phi_2 \in J_2(\mathfrak{g})$ and some constant c, has full holonomy G and is not locally symmetric.

This shows, in particular, that the representations in (1) do occur as holonomy representations, and hence proves Theorem 1.1. Also, the representation in (i) is precisely the representation H_3 which has already been shown to occur as an exotic holonomy in [Br3].

Proof. We shall present the proof for the representations in (ii) only. In this case, the G-invariant symplectic form is

$$\sigma(e_1 \otimes x_1, e_2 \otimes x_2) = <e_1, e_2> (x_1, x_2),$$

where, as before, $<\ ,\ >$ and $(\ ,\ )$ denote the area form on $\mathbb{F}^2$ and the inner product on $\mathbb{F}^n$, respectively, with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We shall identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form B on $\mathfrak{g}$, given by

$$B(A + M, e_1 e_2 + x_1 \wedge x_2) = <Ae_1, e_2> + (Mx_1, x_2) \text{ with } A \in \mathfrak{sl}(\mathbb{F}^2) \text{ and } M \in \mathfrak{so}(\mathbb{F}^n).$$

Here, once again we used the identifications $\mathfrak{sl}(\mathbb{F}^n) \cong S^2(\mathbb{F}^n)$ and $\mathfrak{so}(\mathbb{F}^n) \cong \Lambda^2 \mathbb{F}^n$ from (6). Now from the explicit description of $K(\mathfrak{g})$ in Proposition 2.1 it is straightforward to show that $J_2(\mathfrak{g}) = (\mathfrak{g} \otimes K(\mathfrak{g})) \cap \left(S^2(\mathfrak{g}) \otimes \Lambda^2 V\right)$ is one-dimensional and spanned by the element ϕ_2, given by

$$\begin{aligned} \phi_2(A_1 + M_1, A_2 + M_2, \quad & e_1 \otimes x_1, e_2 \otimes x_2) := \\ & \sigma(e_1 \otimes x_1, e_2 \otimes x_2)(B(A_1 + M_1, A_2 + M_2)) \\ & -(B(A_1, e_1 e_2) B(M_2, x_1 \wedge x_2) + B(A_2, e_1 e_2) B(M_1, x_1 \wedge x_2)) \\ & + <e_1, e_2> ((M_1 x_1, M_2 x_2) + (M_1 x_2, M_2 x_1)). \end{aligned} \tag{17}$$

The isomorphism ρ from (7) then coincides with the map $\phi_2' : \mathfrak{g}^* \to K(\mathfrak{g})$ from (15), again after identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form B. Moreover, by (3) in Remark 2.2, $K_0(\mathfrak{g}) \subseteq K(\mathfrak{g})$ is *open dense*. Then Theorem 3.7 completes the proof. ∎

We can show even more. Namely, surprisingly enough, the converse of Theorem 3.8 is true:

Theorem 3.9 *Let $G \subseteq Gl(V)$ be one of the representations in Theorem 3.8 other than $Sl(2, \mathbb{R})SO(2, \mathbb{R})$. Then all torsion-free affine connections whose holonomy is contained in G are Poisson connections induced by the admissible maps* (16)*.*

This will follow immediately from the next Theorem, together with the explicit description of the spaces $J_2(\mathfrak{g})$ in each case.

Theorem 3.10 *Let $G \subseteq Gl(V)$ and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be as before. Suppose the following conditions are satisfied:*

(i) *G is connected, reductive and acts irreducibly on V,*

(ii) *there is a $\phi_2 \in J_2(\mathfrak{g})^G$ such that the corresponding G-equivariant maps ϕ_2' and ϕ_2'' from* (15) *are isomorphisms.*

Then every torsion-free affine connection whose holonomy is contained in G is a Poisson connection induced by a polynomial map

$$\phi = \phi_2 + \tau,$$

with $\phi_2 \in J_2(\mathfrak{g})$ from above and a (possibly vanishing) G-invariant 2-form $\tau \in \Lambda^2 V^*$.

For the proof, we shall need the following Lemma whose proof will be given in the appendix.

Lemma 3.11 *Let $G \subseteq Gl(V)$ be an irreducible representation of a connected, reductive Lie group G, and let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be the corresponding Lie algebra. If $\tau \in V^* \otimes V^*$ satisfies the condition*

$$\tau(x, A \cdot y) = \tau(y, A \cdot x) \quad \textit{for all } x, y \in V \textit{ and } A \in \mathfrak{g},$$

then τ is skew-symmetric and hence a G-invariant 2-form.

Proof of Theorem. Let $F \subseteq \mathfrak{F}_V$ be a G-structure on the manifold M where $\mathfrak{F}_V \to M$ is the V-valued coframe bundle of M, and denote the tautological V-valued 1-form on F by θ. Suppose that F is equipped with a torsion-free connection, i.e. a $\mathfrak{g}$-valued 1-form ω on F. Since ϕ_2' is an isomorphism, the *first and second structure equations* read

$$\begin{aligned} d\theta &= -\omega \wedge \theta \\ d\omega &= -\omega \wedge \omega - 2(\phi_2'(\mathbf{a})) \circ (\theta \wedge \theta), \end{aligned} \tag{18}$$

where $\mathbf{a} : F \to \mathfrak{g}^*$ is a G-equivariant map. Differentiating (18) and using that ϕ_2'' is an isomorphism yields the *third structure equation* for the differential of $\mathbf{a}$:

$$d\mathbf{a} = -\omega \cdot \mathbf{a} + \jmath(\mathbf{b} \otimes \theta), \tag{19}$$

for some G-equivariant map $\mathbf{b} : F \to V^*$, where $\jmath : V^* \otimes V \to \mathfrak{g}^*$ is the natural projection. The multiplication in the first term refers to the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. In other words, (19) should be read as

$$\begin{aligned} (\xi_A \mathbf{a})(B) &= \mathbf{a}([A, B]) \\ (\xi_x \mathbf{a})(B) &= \mathbf{b}(B \cdot x). \end{aligned}$$

Let us define the map $\mathbf{c} : F \to V^* \otimes V^*$ by

$$\mathbf{c}_p(x, y) := d\mathbf{b}(\xi_x)(y) - \phi_2(\mathbf{a}_p, \mathbf{a}_p, x, y). \tag{20}$$

Differentiation of (19) yields

$$\mathbf{c}_p(x, Ay) = \mathbf{c}_p(y, Ax) \quad \text{for all } x, y \in V \text{ and all } A \in \mathfrak{g}. \tag{21}$$

Then by (i), (21) and Lemma 3.11 we conclude that $\mathbf{c}_p \in \Lambda^2 V^*$ is G-invariant. Moreover, differentiation of (20) implies that $\xi_A(\mathbf{c}) = 0$ and $(\xi_x \mathbf{c})(y, z) = (\xi_y \mathbf{c})(x, z)$ for all $A \in \mathfrak{g}$ and $x, y, z \in V$. Since $\mathbf{c}$ is skew-symmetric, it follows that

$$d\mathbf{c} = 0,$$

i.e. $\mathbf{c}_p \equiv \tau \in \Lambda^2 V^*$ is *constant*. Thus, the G-equivariance of $\mathbf{b}$ and (20) yield

$$d\mathbf{b} = -\omega \cdot \mathbf{b} + \left(\mathbf{a}_p^2 \lrcorner \phi_2 + \tau\right) \circ \theta, \tag{22}$$

where $\lrcorner$ refers to the contraction of $\mathbf{a}_p^2 \in S^2(\mathfrak{g}^*)$ with $\phi_2 \in S^2(\mathfrak{g}) \otimes \Lambda^2 V^*$. In other words, (22) should be read as

$$\begin{array}{rcl}(\xi_A \mathbf{b})(y) & = & \mathbf{b}(A \cdot y) \\ (\xi_x \mathbf{b})_p(y) & = & \phi_2(\mathbf{a}_p, \mathbf{a}_p, x, y) + \tau(x, y).\end{array}$$

Let us now define the Poisson structure on $W^* = \mathfrak{g}^* \oplus V^*$ induced by $\phi := \phi_2 + \tau$, and let $\pi := \mathbf{a} + \mathbf{b} : F \to W^*$. From (19) and (22) it follows that $\pi_*(\xi_w) = \eta_w$ for all $w \in W$, and from there one can show that, at least locally, the connection is indeed a Poisson connection induced by ϕ. ∎

From the complete characterization in Theorem 3.9, we can deduce the following properties which summarize our discussion so far:

Corollary 3.12 *Let M be a manifold which carries a torsion-free connection whose holonomy is contained in one of the groups $G \subseteq Gl(V)$ from Theorem 3.8 other than $Sl(2,\mathbb{R})SO(2,\mathbb{R})$, and let $\phi = \phi_2 + c\sigma$ be the admissible map which induces this connection. Then we have the following.*

(1) *The connection is analytic.*

(2) *The map $\pi := \mathbf{a} + \mathbf{b} : F \to W^*$ has constant even rank $2k$ which we shall call the rank of the connection. $k = 0$ iff the connection is flat.*

(3) *$\pi(F)$ is contained in a $2k$-dimensional characteristic leaf Σ of the Poisson structure on W^* induced by ϕ. In particular, $\pi : F \to \Sigma$ is a submersion onto its image.*

(4) *Conversely, given a characteristic leaf $\Sigma \subseteq W^*$ of maximal rank, then Σ can be covered by open neighborhoods $\{U_\alpha\}$ such that there are Poisson connections with $\pi(F_\alpha) = U_\alpha$.*

(5) *Let $\mathfrak{s} \subseteq \mathfrak{X}(F)$ be the Lie algebra of infinitesimal symmetries of the connection, i.e. those vector fields whose flows preserve the connection. Then $dim(\mathfrak{s}) = dim(W^*) - 2k$. In particular, if $n \equiv 0, 1 \mod 4$ then $dim(\mathfrak{s}) > 0$.*

(6) *The moduli space of torsion-free connections with any of the above holonomies is finite dimensional. In particular, the 2nd derivative of the curvature at a single point in M completely determines the connection on all of M.*

Proof. (2) and (3) follow from the identity $\pi_*(\xi_w) = \eta_w$. (1) and (4) are clear from the construction of the Poisson connections and the analyticity of ϕ, whereas (6) follows from the structure equations in the proof of Theorem 3.10.

To show (5), let $f : W^* \to \mathbb{F}$ be a function which vanishes on $\Sigma = \pi(F)$. Then it is easy to see that $\#\pi^*(df)$ is an infinitesimal symmetry. It follows that $dim(\mathfrak{s}) \geq dim(W^*) - 2k$. On the other hand, if $X \in \mathfrak{s}$ then $\pi_*(X) = 0$, hence $dim(\mathfrak{s}) \leq dim(W^*) - 2k$. ∎

Of course, (4) is not an optimal statement. One would like to show that there are connections such that $\pi(F)$ is an *entire characteristic leaf.* Also, our method does not prove the existence of connections whose rank is not maximal. The difficulty is that, in general, one cannot expect to have a *global* symplectic realization $\pi : S \to W^*$. In fact, even if we restrict to the subset $W^{reg} \subseteq W^*$ where the Poisson structure has maximal rank, then the obstruction for the existence of a global symplectic realization is given by a class in $H^3_{rel}(W^{reg}, \mathcal{F})$, where $\mathcal{F}$ is the foliation by symplectic leafs [V].

Also, (5) is not an optimal statement either. Namely, as was shown in [CS], if $n = 3$ then the dimension of the Lie algebra $\mathfrak{s}$ of infinitesimal symmetries is at least 2. In fact, it seems likely that $dim(\mathfrak{s}) > 0$ in *any* dimension, even though we do not have a proof of this assertion.

For H_3-connections, all statements in Corollary 3.12 have been shown in [Br3] by MAPLE calculations. The same kind of calculations was used in [CS] to prove Corollary 3.12 for $n = 3$. Our current approach, however, seems more conceptual. In fact, it was a deeper understanding of the structure of H_3-connections which led us to the construction of Poisson connections presented in this paper.

There is another global result whose analogue for H_3-connections has been demonstrated in [Sc]:

Theorem 3.13 *Let M be a manifold which carries a torsion-free connection whose holonomy is contained in one of the groups $G \subseteq Gl(V)$ from* (1) *other than $Sl(2,\mathbb{R})SO(n,\mathbb{R})$. Then M is geodesically incomplete, unless the connection is locally symmetric. If the latter is the case, then the holonomy is a proper subgroup of G.*

Proof. The holonomy of a locally symmetric connection must leave the curvature tensor invariant. Since $K(\mathfrak{g})$ does not contain any non-zero G-invariant element, the last statement follows.

Throughout this proof, we shall use the same notations and identifications as in the proof of Theorem 3.8. In particular, whenever it is convenient we shall regard $\mathbf{a}$ as a $\mathfrak{g}$-valued form, and write $\mathbf{a}_p = A_p + M_p$ with $A_p \in \mathfrak{sl}(2,\mathbb{F})$ and $M_p \in \mathfrak{so}(n,\mathbb{F})$.

M is geodesically complete iff the vector fields $\xi_x \in \mathfrak{X}(F)$ are complete for all $x \in V$, and we shall assume this from now on. To prove the Theorem, we have to show that the connection is locally symmetric, which is the case iff $\mathbf{b} \equiv 0$. If $M_p \equiv 0$ then (19) implies that $\mathbf{b} \equiv 0$. Thus, we assume that at some $p_0 \in F$, $M_{p_0} \neq 0$ and $\mathbf{b}_{p_0} \neq 0$ and shall deduce a contradiction.

Let $\mathcal{N} := \{y \in \mathbb{F}^n \mid (y, y) = 0\}$. By the indefiniteness of $(\ ,\ )$, $\mathcal{N}$ spans all of $\mathbb{F}^n$. Therefore, $\mathbf{b}_{p_0}(e_1 \otimes y) \neq 0$ for some $e_1 \in \mathbb{F}^2, y \in \mathcal{N}$. Let $\nu := (M_{p_0} \cdot y, M_{p_0} \cdot y)$. If $\mathbb{F} = \mathbb{R}$ and $(\ ,\ )$

has signature $(1, n-1)$, then it follows that $\nu \leq 0$. Therefore, after possibly replacing $(\ ,\ )$ by its negative, we may assume that $\nu \geq 0$ and, if $\mathbb{F} = \mathbb{R}$, the signature is (p, q) with $p > 1$. It follows that there is a basis $\{x_1, \ldots, x_{n-2}, y, z\}$ of $\mathbb{F}^n$ and $e_1, e_2 \in \mathbb{F}^2$ such that the following hold:

$$\begin{gathered} (x_i, x_j) = \delta_i^j \varepsilon_i \ \text{ with } \ \varepsilon_i = \pm 1, \quad \varepsilon_1 = 1, \\ (x_i, y) = (x_i, z) = (y, y) = (z, z) = 0, \quad (y, z) = 1, \\ <e_1, e_2>= 1, \quad (M_{p_0} \cdot y, M_{p_0} \cdot y) \geq 0, \ \text{ and } \ \mathbf{b}_{p_0}(e_1 \otimes y) \neq 0. \end{gathered} \tag{23}$$

Now, let us define

$$\xi_1 := \xi_{e_1 \otimes x_1}, \quad \xi_2 := \xi_{e_2 \otimes y}, \quad A_0 := -e_1^2 \quad \text{and} \quad M_0 := 2y \wedge x_1,$$

and the functions

$$f_1(p) := \mathbf{a}_p(A_0), \quad f_2(p) := \mathbf{a}_p(M_0) \quad \text{and} \quad g(p) := -2\mathbf{b}_p(e_1 \otimes y).$$

From (17), (19), (22) and (23) we calculate

$$\begin{gathered} \xi_1(f_1) = \xi_2(f_2) = 0, \quad \xi_1(f_2) = \xi_2(f_1) = g, \\ \xi_1(g) = 2f_1 f_2, \ \text{ and } \ \xi_2(g) = 4(M_p \cdot y, M_p \cdot y). \end{gathered} \tag{24}$$

For $v \in \mathbb{F}^n$, we have $(v, v) = \sum_i \varepsilon_i (v, x_i)^2 + (v, y)(v, z)$. Thus,

$$(M_p \cdot y, M_p \cdot y) = \sum_i \varepsilon_i (M_p \cdot y, x_i)^2 = \sum_i \varepsilon_i (\mathbf{a}_p(y \wedge x_i))^2.$$

We have $\mathbf{a}_p(y \wedge x_1) = B(M_p, y \wedge x_1) = \frac{1}{2} f_2(p)$, and from (19) and (23) we get that $\xi_k(\mathbf{a}_p(y \wedge x_i)) = \mathbf{b}_p((y \wedge x_i) \cdot \xi_k) = 0$ for $i > 1$. Therefore,

$$\xi_k\left(\xi_2(g) - f_2^2\right) = 0 \quad \text{for } k = 1, 2. \tag{25}$$

For arbitrary constants $0 \neq c_1, c_2 \in \mathbb{F}$, define the vector field ξ and the function f by

$$\xi := c_1 \xi_1 + c_2 \xi_2 \ \text{ and } \ f := c_1^2 f_1 + 2c_1 c_2 f_2.$$

From (24) we compute that

$$\begin{aligned} \xi^2(f) &= 3c_1^2 c_2 \xi(g) \\ &= 3c_1^2 c_2 (2c_1 f_1 f_2 + c_2 \xi_2(g)) \\ &= f^2 - c_1^2 \left((c_1 f_1 - c_2 f_2)^2 - 3c_2^2(\xi_2(g) - f_2^2)\right). \end{aligned}$$

By (25), $\xi\left((c_1 f_1 - c_2 f_2)^2 - 3c_2^2(\xi_2(g) - f_2^2)\right) = 0$. It follows that along the flow line of ξ, f satisfies the differential equation

$$y'' = y^2 + C, \tag{26}$$

where C is a constant. By the assumption of completeness, f must be a *global* solution of (26). The following Lemma will be shown in the appendix.

Lemma 3.14 (i) *The only holomorphic solutions of* (26) *which are defined on all of* $\mathbb{C}$ *are constants.*

(ii) *Suppose there is a real solution of* (26) *which is defined on all of* $\mathbb{R}$*. If for* $x_0 \in \mathbb{R}$, $y(x_0) > 0$ *and* $y'(x_0) \neq 0$*, then* $y''(x_0) < 0$.

We shall use this Lemma to get the desired contradiction. If $\mathbb{F} = \mathbb{C}$, the Lemma implies that $3c_1^2 c_2 g = \xi(f) \equiv 0$, contradicting $g(p_0) = -2\mathbf{b}_{p_0}(e_1 \otimes y) \neq 0$.

Consider now the case $\mathbb{F} = \mathbb{R}$. From (24) and (25) we have $(\xi_2)^3(f_1) = 0$. Moreover, $(\xi_2)^2(f_1)_{p_0} = 4(M_{p_0} \cdot y, M_{p_0} \cdot y) \geq 0$ and $\xi_2(f_1)_{p_0} = g(p_0) \neq 0$. Therefore, since ξ_2 is complete, there is a point q_0 on the flow line of ξ_2 through p_0 which, in addition to (23), also satisfies $f_1(q_0) > 0$. W.l.o.g. we assume that $q_0 = p_0$.

Now $y(x_0) = f(p_0) = (c_1^2 f_1 + 2c_1 c_2 f_2)(p_0)$ and $y''(x_0) = \xi^2(f)_{p_0} = 3c_1^2 c_2 (2c_1 f_1 f_2 + c_2 \xi_2(g))(p_0)$. Since $f_1(p_0) > 0$ and $\xi_2(g)_{p_0} \geq 0$ it follows that for a suitable choice of c_1, c_2, both $y(x_0) > 0$ and $y''(x_0) \geq 0$ can be achieved. But $y'(x_0) = g(p_0) \neq 0$, hence again, we get a contradiction from the Lemma. ∎

It is now clear that Theorem 1.2 is simply a restatement of parts of Corollary 3.12 and Theorem 3.13.

4 Appendix

We shall now give the proof of two technical Lemmas.

Lemma 3.11 *Let* $G \subseteq Gl(V)$ *be an irreducible representation of a connected, reductive Lie group* G*, and let* $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ *be the corresponding Lie algebra. If* $\tau \in V^* \otimes V^*$ *satisfies the condition*

$$\tau(x, A \cdot y) = \tau(y, A \cdot x) \quad \text{for all } x, y \in V \text{ and } A \in \mathfrak{g},$$

then τ *is skew-symmetric and hence a* G*-invariant 2-form.*

Proof. Clearly, the Lemma is invariant under complexification, hence we may assume that $G, \mathfrak{g}$ and V are complex. Also, if $\mathfrak{g}_s \subseteq \mathfrak{g}$ is the semi-simple part of $\mathfrak{g}$, then $\mathfrak{g}_s$ acts irreducibly on V as well. Thus, we may assume w.l.o.g. that $\mathfrak{g}$ is semi-simple. Let

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \quad \text{and} \quad V = \bigoplus_{\lambda \in \Lambda} V_\lambda$$

be a Cartan and weight space decomposition of $\mathfrak{g}$ and V, respectively. We shall denote elements of $\mathfrak{t}$, $\mathfrak{g}_\alpha$ and V_λ by A_0, A_α and x_λ, respectively. Also, we always write $\alpha, \beta, \ldots$ for roots, whereas $\lambda, \mu, \ldots$ stand for weights. Let $\lambda_{max} \in \mathfrak{t}^*$ be the maximal weight, and $x_{max} \in V_{\lambda_{max}}$ be a highest weight vector.

The proof now proceeds as follows. Given τ as above, we shall prove:

Step 1: if $\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{C})$ and $\lambda + \mu \neq 0$, then $\tau(x_\lambda, x_\mu) = 0$.
Step 2: $\tau(x_\lambda, x_\mu)\mu = \tau(x_\mu, x_\lambda)\lambda$ for all $\lambda, \mu \in \Lambda$. In particular, $\tau(x_\lambda, x_\mu) = 0$ if λ, μ are linearly independent.
Step 3: if λ, μ are scalar multiples of λ_{max} and $\lambda, \mu, \lambda + \mu \neq 0$ then $\tau(x_\lambda, x_\mu) = \tau(x_\mu, x_\lambda) = 0$.
Step 4: $\tau(x_{max}, x_0) = \tau(x_0, x_{max}) = 0$.
Step 5: τ is skew-symmetric.

Proof of step 1: Suppose $\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{C})$, and $V \cong V_n$ is the (unique) irreducible $n+1$-dimensional representation of $\mathfrak{g}$. Then it is an easy exercise to show that the only elements $\tau \in V^* \otimes V^*$ which satisfy the above identity are $\tau = 0$ if n is even, and a multiple of the symplectic $\mathfrak{g}$-invariant 2-form on V if n is odd. From this, step 1 follows. The proof is omitted.

Proof of step 2: $\tau(x_\lambda, A_0 \cdot x_\mu) = \tau(x_\mu, A_0 \cdot x_\lambda)$, hence $\mu(A_0)\tau(x_\lambda, x_\mu) = \lambda(A_0)\tau(x_\mu, x_\lambda)$ for all $A_0 \in \mathfrak{t}$. This proves step 2.

Proof of step 3: Let λ, μ be as above, and let $\alpha \in \Delta$ be linearly independent of λ_{max}. We compute $\tau(x_\lambda, A_\alpha A_{-\alpha} \cdot x_\mu) = \tau(A_{-\alpha} \cdot x_\mu, A_\alpha \cdot x_\lambda) = 0$ by step 2; indeed, $A_{-\alpha} \cdot x_\mu \in V_{\mu-\alpha}$, $A_\alpha \cdot x_\lambda \in V_{\lambda+\alpha}$, and $\mu - \alpha$, $\lambda + \alpha$ are linearly independent if $\lambda + \mu \neq 0$. Likewise, $\tau(x_\lambda, A_{-\alpha} A_\alpha \cdot x_\mu) = 0$, and hence $0 = \tau(x_\lambda, [A_\alpha, A_{-\alpha}] \cdot x_\mu) = \mu([A_\alpha, A_{-\alpha}])\tau(x_\lambda, x_\mu)$. Thus, we have $\lambda_{max}([A_\alpha, A_{-\alpha}])\tau(x_\lambda, x_\mu) = 0$ and similarly, $\lambda_{max}([A_\alpha, A_{-\alpha}])\tau(x_\mu, x_\lambda) = 0$. But if $\lambda_{max}([A_\alpha, A_{-\alpha}]) = 0$ for all roots α independent of λ_{max}, then it follows that $\mathfrak{g}$ must contain $\mathfrak{sl}(2,\mathbb{C})$ as a summand, and λ_{max} is a multiple of the root corresponding to this summand.

In this case, however, since the representation is faithful, it must be equivalent to the action of $\mathfrak{sl}(2,\mathbb{C})$ on V_n. This observation together with step 1 implies step 3.

Proof of step 4: The equation $\tau(x_0, x_{max}) = 0$ follows immediately from step 2.

Since the representation is irreducible, we have $V_0 = span\{A_{-\alpha} \cdot V_\alpha \mid \alpha \in \Delta\}$. Thus, in order to show step 4, we need to show

$$\tau(x_{max}, A_{-\alpha} \cdot x_\alpha) = 0 \text{ for all } \alpha \in \Delta, x_\alpha \in V_\alpha. \tag{27}$$

If $\alpha \neq \lambda_{max}$ we have $\tau(x_{max}, A_{-\alpha} \cdot x_\alpha) = \tau(x_\alpha, A_{-\alpha} \cdot x_{max}) = 0$ by steps 2 and 3; indeed, $A_{-\alpha} \cdot x_{max} \in V_{\lambda_{max}-\alpha}$ and $\lambda_{max} - \alpha \notin \{0, -\alpha\}$. Thus, (27) follows in this case.

Next, suppose that $\alpha = \lambda_{max}$, and let β and $x_{\alpha+\beta}$ be such that $x_{max} = A_{-\beta} \cdot x_{\alpha+\beta}$. By irreducibility, this is always possible. Note that $\beta \neq \alpha$, since by maximality of $\alpha = \lambda_{max}$, 2α is not a root. Then

$$\begin{aligned}
\tau(x_{max}, A_{-\alpha} \cdot x_\alpha) &= \tau(x_\alpha, A_{-\alpha} A_{-\beta} \cdot x_{\alpha+\beta}) \\
&= \tau(x_\alpha, ([A_{-\alpha}, A_{-\beta}] + A_{-\beta} A_{-\alpha}) \cdot x_{\alpha+\beta}) \\
&= \tau(x_{\alpha+\beta}, [A_{-\alpha}, A_{-\beta}] \cdot x_\alpha) + \tau(A_{-\alpha} \cdot x_{\alpha+\beta}, A_{-\beta} \cdot x_\alpha) \\
&:= I + II.
\end{aligned}$$

Now $x_{\alpha+\beta} \in V_{\alpha+\beta}$ and $[A_{-\alpha}, A_{-\beta}] \cdot x_\alpha \in V_{-\beta}$. Thus, by step 2 and the fact that $\alpha \neq \beta$ it follows that $I = 0$. For II, we have $A_{-\alpha} \cdot x_{\alpha+\beta} \in V_\beta$ and $A_{-\beta} \cdot x_\alpha \in V_{\alpha-\beta}$. Now if β is independent of α, step 2 implies that $II = 0$. If $\beta = -\alpha$ then $A_{-\beta} \cdot x_\alpha \in V_{\alpha-\beta} = V_{2\alpha} = 0$ by maximality of $\lambda_{max} = \alpha$, and again, $II = 0$ follows. Thus, (27) is shown in this case as well, and step 4 follows.

Proof of step 5: Let $\tau^{sym}(x, y) := \tau(x, y) + \tau(y, x)$. From steps 2 through 4, it follows that $\lambda_{max} + \lambda \neq 0$ implies that $\tau(x_{max}, x_\lambda) = \tau(x_\lambda, x_{max}) = 0$. Also, if $x_{-max} \in V_{-\lambda_{max}}$, then by step 2, $\tau^{sym}(x_{max}, x_{-max}) = 0$.

Thus, we conclude that $\tau^{sym}(x_{max}, \_) = 0$. But this is true for *any* choice of Cartan decomposition. Exploiting all possible decompositions, we conclude

$$\tau^{sym}(g \cdot x_{max}, \_) = 0 \text{ for all } g \in G.$$

By irreducibility, the G-orbit of x_{max} spans all of V, and from there it follows that $\tau^{sym} = 0$, hence τ is skew-symmetric. ∎

Lemma 3.14 *Consider the differential equation $y'' = y^2 + C$ (26) for a constant C.*

(i) *The only holomorphic solutions of* (26) *which are defined on all of $\mathbb{C}$ are constants.*

(ii) *Suppose there is a real solution of* (26) *which is defined on all of $\mathbb{R}$. If for $x_0 \in \mathbb{R}$, $y(x_0) > 0$ and $y'(x_0) \neq 0$, then $y''(x_0) < 0$.*

Proof. First of all, we integrate (26) to

$$(y')^2 = \frac{2}{3}y^3 + 2Cy + C_1 \tag{28}$$

for some constant C_1.

(i) Let y be a holomorphic solution which is an entire function. Define

$$\begin{array}{rccc} \phi: & \mathbb{C} & \longrightarrow & \mathbb{CP}_2 \\ & x & \longmapsto & [y(x) : y'(x) : 1]. \end{array}$$

By (28), the image of ϕ is contained in the cubic $\mathcal{C}$ given by $3q^2r - 2p^3 - 6Cpr^2 - 3C_1r^3 = 0$, where p, q, r are the coordinates in $\mathbb{CP}_2$.

We calculate that $\mathcal{C}$ is a regular curve of genus 1 if $16C^3 + 9C_1^2 \neq 0$; it is a rational curve with a node if $16C^3 + 9C_1^2 = 0$ and $C \neq 0$, and it is a rational curve with a cusp if $C = C_1 = 0$.

Since $im(\phi)$ does not contain the point $[0 : 1 : 0] \in \mathcal{C}$, it follows that in the first case ϕ maps $\mathbb{C}$ non-surjectively to a genus 1 curve. By Picard's Theorem, this implies that ϕ, and hence y, is constant.

In the second case, we find a parametrization of $\mathcal{C}$ and compute that there must be a function $t = t(x)$ such that

$$y = \tfrac{3}{2C}\left(C_1 - 4C^2t^2\right) \quad \text{and} \quad y' = \tfrac{2C}{C_1}\left(3C_1 - 8C^2t^2\right)t.$$

But this implies that either $t \equiv 0$ or $t' = \frac{4C^2}{3C_1}t^2 - \frac{1}{2}$. It is straightforward to verify that the only global solutions of the latter equation are constants.

In the third case $C = C_1 = 0$, a parametrization is given by $y = 6t^2, y' = 12t^3$ for some entire function t. But then $12tt' = 12t^3$, which again has no global non-constant solution.

(ii) Consider a global real solution y and $x_0 \in \mathbb{R}$ as above. After possibly replacing $y(x)$ by the solution $y(-x)$ and x_0 by $-x_0$, we may assume that $y'(x_0) > 0$. If $y^2(x_0) + C = y''(x_0) \geq 0$, then elementary calculus shows that $\lim_{x\to\infty} y = \infty$. Thus, from (28), we have for sufficiently large x that $(y')^2 > \frac{1}{3}y^3$, and hence

$$\left(y^{-\frac{1}{2}}\right)' = -\frac{1}{2}y^{-\frac{3}{2}}y' < -\frac{1}{2\sqrt{3}},$$

contradicting $y^{-\frac{1}{2}} > 0$ for large x. Thus, $y''(x_0) \geq 0$ is impossible. ∎

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Quo-Shin Chi — Department of Mathematics, Campus Box 1146, Washington University, St. Louis, Mo 63110, USA
chi@artsci.wustl.edu

Sergey A. Merkulov — Department of Pure Mathematics, Glasgow University, 15 University Gardens, Glasgow G12 8QW, UK
sm@maths.glasgow.ac.uk

Lorenz J. Schwachhöfer — Max-Planck-Institut für Mathematik, Gottfried-Claren-Strasse 26, 53225 Bonn, Germany
lorenz@mpim-bonn.mpg.de